

Rotation of quantum liquid without singular vortex lines

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Abstract.

The operator equations for quantum hydrodynamics are discussed and solved in a simple cylindrical geometry. We find a solution with the velocity curl “frozen” into a density of the liquid in the absence of singular vortex lines. The spectrum of small oscillations around this solution is found as a generalization of the standard phonon spectrum, and the stability of the solution is demonstrated.

1. Introduction

With the development of physics of cooled atoms in traps, the studies of quantum rotation in a many-body system have revived and entered a new stage. In superfluid helium-4, as it is well known [1, 2], the vortices with quantized circulation emerge at a certain critical cranking velocity. At sufficiently fast rotation, the vortex lines form a lattice that on average imitates the rigid-body rotation [3]. Analogous vortices appear in penetration of the magnetic field into superconductors of second kind [4]. Roughly speaking, the first vortex can be born by a perturbation violating the angular momentum conservation at the angular velocity corresponding to the crossing of single-particle levels corresponding to the zero and non-zero angular momentum per particle. A nuclear analog is the so-called back-bending that occurs also at a band crossing in the rotating frame [5]. Similar phenomena are known for molecules [6], predicted for neutron stars [7], and important for some quantum field-theoretical models [8].

Our study is also partly motivated by the recent explosion of activity in experimental and theoretical studies of superfluidity and other quantum phenomena in trapped ultracold atomic systems [9, 10, 11] and quantum Hall superfluids [12, 13]. Also, there have been many experimental studies of the flow properties of superfluids confined in porous media such as Vycor glass and containers packed with fine powder [14, 15, 16, 17].

The question of a possibility for a quantum liquid to be in a state of rotational excitation without forming individual vortices or a vortex lattice is still debatable. Such rotation should have a moment of inertia close to that of a rigid body. A small quantum system, such as a medium or heavy atomic nucleus, clearly displays numerous rotational bands built on various intrinsic configurations. A Fermi system without pairing correlations should have, with semiclassical accuracy, the moment of inertia corresponding to the rigid-body rotation for a given density distribution [18]. In agreement with the substantial presence of pairing, the nuclear moment of inertia is considerably smaller than that of a rigid body but still much greater than that of a perfect liquid [19, 20]. Rotation of an atomic system with and without a quantized vortex was considered in [21] (see also references therein). The evidence for the so-called Landau state with identically vanishing curl of the superfluid velocity was presented in [22]. Especially rich spectrum of rotational phases is predicted and partially observed in superfluid helium-3 [23, 24], where the orbital momentum is strongly coupled with spin structures. New types of rotation are expected at high pressure [25, 26] and for crystallizing quantum liquids [27]. It is also worthwhile to mention here the fractional quantum Hall effect where the vorticity of Laughlin's state is proportional to the local density of electronic fluid [28, 29].

The statement that a quantum liquid rotates necessarily with emergence of quantized vortex lines seems to be too restricting.

In the present article we consider the quantum hydrodynamics in the original form developed by Landau [30]. With the exact solution of quantum operator equations of motion we come to the type of rotation that is not related to the formation of singular vortex lines (the vortex solution is possible as well). We show also that the new solution is stable with respect to small oscillations and find explicitly the spectrum of such oscillatory modes.

2. Quantum hydrodynamics in operator formulation

The equations of quantum hydrodynamics for a system of identical Bose-particles of mass m were formulated by Landau in terms of the field operators of density, $\rho(\mathbf{x}) = mn(\mathbf{x})$, and current density $\mathbf{j}(\mathbf{x})$,

$$n(\mathbf{x}) = \sum_a \delta(\mathbf{r}_a - \mathbf{x}), \quad (1)$$

$$\mathbf{j}(\mathbf{x}) = \frac{1}{2m} \sum_a [\mathbf{p}_a, \delta(\mathbf{r}_a - \mathbf{x})]_+, \quad (2)$$

where the subscript a labels single-particle variables, and $[\dots]_+$ means the anticommutator. The quantization is performed through the commutation relations,

$$[n(\mathbf{x}), n(\mathbf{x}')] = 0, \quad (3)$$

$$[\mathbf{j}(\mathbf{x}), n(\mathbf{x}')] = -i \frac{\hbar}{m} n(\mathbf{x}) \nabla_{\mathbf{x}} \delta(\mathbf{x} - \mathbf{x}'), \quad (4)$$

and

$$[j_i(\mathbf{x}), j_j(\mathbf{x}')] = i \frac{\hbar}{m} \varepsilon_{ijk} (\text{curl } \mathbf{j}(\mathbf{x}))_k \delta(\mathbf{x} - \mathbf{x}'). \quad (5)$$

The microscopic velocity field $\mathbf{v}(\mathbf{x})$ can be defined (at the points where the density n differs from zero) as

$$\mathbf{v}(\mathbf{x}) = \frac{1}{2} \left[\frac{1}{n(\mathbf{x})}, \mathbf{j}(\mathbf{x}) \right]_+. \quad (6)$$

It satisfies the commutation relations

$$[\mathbf{v}(\mathbf{x}), n(\mathbf{x}')] = -i \frac{\hbar}{m} \nabla_{\mathbf{x}} \delta(\mathbf{x} - \mathbf{x}'), \quad (7)$$

and

$$[v_i(\mathbf{x}), v_j(\mathbf{x}')] = i \frac{\hbar}{m} \varepsilon_{ijk} \frac{1}{n(\mathbf{x})} (\text{curl } \mathbf{v}(\mathbf{x}))_k \delta(\mathbf{x} - \mathbf{x}'). \quad (8)$$

As a consequence of eq. (7), the density operator commutes with the curl of velocity,

$$[n(\mathbf{x}), \text{curl } \mathbf{v}(\mathbf{x}')] = 0. \quad (9)$$

The whole algebra can be simply postulated or derived in terms of the underlying particle variables, either in configuration space [30] (the right hand side of eq. (8) is given there with an opposite sign), or in the secondary quantized form [31].

There are statements in the literature, for example [32, 33], that quantum hydrodynamics formulated in such a way contains intrinsic contradictions. Indeed, with formal manipulations by commutation relations, one can construct the states, where the expectation value of $n(\mathbf{x})$ is negative and very large by absolute magnitude. However, this is not a physical difficulty. Thinking of a real quantum liquid, we should have in mind the Hamiltonian of the type

$$H = \frac{m}{2} \int d^3x \mathbf{v}(\mathbf{x}) n(\mathbf{x}) \mathbf{v}(\mathbf{x}) + W\{n\}, \quad (10)$$

where the functional W depends on density and its spatial gradients. Since the liquid consists of quantum particles, this functional contains the well known quantum potential W_q ,

$$W = W_q + \overline{W}, \quad W_q = \frac{\hbar^2}{8m} \int d^3x \frac{(\nabla n)^2}{n}. \quad (11)$$

Similarly to the centrifugal barrier in the radial Schrödinger equation, the term W_q does not allow the system to penetrate into the regions of $n < 0$. Such “fall onto the center” is possible only if the interaction \overline{W} has a stronger singularity at $n \rightarrow 0$. We can conclude that the growth of quantum fluctuations at $n \rightarrow 0$ prevents the transition into unphysical states.

Being interested in the spectrum of low-lying elementary excitations in superfluid ^4He (phonons and rotons), Landau considered in Ref. [30] only irrotational motion, still mentioning the possibility of vortex motion in some quantum states which are to be separated by an energy gap from the ground state. Geilikman [31] linearized the equations of motion for weak excitations above the uniform background ($\mathbf{v}^\circ = 0$, $n^\circ =$

const), found the well known Bogoliubov phonon spectrum [34], and showed that the modes with $\text{curl } \mathbf{v} \neq 0$ appear only in higher orders as a result of phonon anharmonicity. Below we construct a class of solutions, where both, classical (c -number) and quantum (operator), parts of $\text{curl } \mathbf{v}$ are present.

3. “Frozen” rotation

Let us define an axial vector operator $\vec{\Gamma}(\mathbf{x})$ that, according to eq. (9), commutes with $n(\mathbf{x})$,

$$\text{curl } \mathbf{v}(\mathbf{x}) = n(\mathbf{x})\vec{\Gamma}(\mathbf{x}). \quad (12)$$

Using our operator algebra, we can derive the commutation relations of the components Γ_i among themselves and with components of velocity, v_j . The simplest non-trivial solutions compatible with the operator algebra correspond to the states of the “frozen” rotation [35], when $\vec{\Gamma}$ is a constant c -number vector, let say along the z -axis, while the density and velocity fields do not depend on z . In this case

$$[\text{curl } \mathbf{v}(\mathbf{x}), H] = -\frac{i\hbar}{2} \vec{\Gamma} \text{div}[\mathbf{v}(\mathbf{x})n(\mathbf{x}) + n(\mathbf{x})\mathbf{v}(\mathbf{x})] = i\hbar\vec{\Gamma} \frac{\partial n(\mathbf{x})}{\partial t}. \quad (13)$$

Therefore, for time-independent density, the quantity $\text{curl } \mathbf{v}(\mathbf{x})$ is the constant of motion that characterizes the stationary states of the system.

We consider a planar motion $\mathbf{v}(\mathbf{r})$ in a circular cylinder of radius R with longitudinal axis z , $v_z = 0$. The transverse components of the velocity field satisfy, according to eqs. (8) and (12), the commutation relations similar to those in a magnetic field,

$$[v_x(\mathbf{r}), v_y(\mathbf{r}')] = \frac{i\hbar}{m} \Gamma \delta(\mathbf{r} - \mathbf{r}'), \quad (14)$$

where $\mathbf{r} = (r, \varphi)$ is the coordinate vector in the (xy) -plane. The operator equations of motion for the velocity field, on this class of states, are given by

$$[v_i(\mathbf{r}), H] = -\frac{i\hbar}{m} \frac{\partial}{\partial x_i} \left\{ \frac{mv^2(\mathbf{r})}{2} + \frac{\delta W}{\delta n(\mathbf{r})} \right\} + i\hbar \Gamma \epsilon_{ijz} \frac{1}{2} [n(\mathbf{r}), v_j(\mathbf{r})]_+. \quad (15)$$

Here the density $n(\mathbf{r})$ has to be expressed in terms of $\text{curl } \mathbf{v}$ due to eq. (12).

We will look for the solution of the set (14,15) as small oscillations above the background characterized by the classical fields of velocity, $\mathbf{v}^\circ(\mathbf{r})$, and density, determined by $\text{curl}_z \mathbf{v}^\circ(\mathbf{r}) = \Gamma n^\circ(\mathbf{r})$. We limit ourselves by the axially symmetric case,

$$v_\varphi^\circ(\mathbf{r}) = u(r), \quad v_r^\circ = 0, \quad n^\circ(\mathbf{r}) = \nu(r). \quad (16)$$

Of course, the angular momentum of the system will be directed along the axis but, due to the azimuthal symmetry, there is no need to make a transformation to a rotating frame and introduce the Lagrangian multiplier of the angular velocity. The similar situation is known also in nuclear physics where the rotational motion around the symmetry axis does not require the cranking procedure. The difference here is that we find collective excited states instead of the more or less random sequence of single-particle excitations

carrying the angular momentum projection onto the symmetry axis in a nucleus rotating around the same axis.

The background state, as follows from eq. (15), should satisfy the equilibrium condition,

$$\frac{mu^2}{r} = \frac{d}{dr} \left(\frac{\delta W}{\delta n} \right)_{n=\nu(r)}. \quad (17)$$

In our geometry,

$$\frac{du}{dr} = -\frac{u}{r} + \Gamma\nu(r). \quad (18)$$

This leads to the general form of the background velocity field,

$$u(r) = \frac{1}{r} \left[K + \Gamma \int_0^r dr' r' \nu(r') \right] = \frac{1}{r} \left[K + \frac{\Gamma}{2\pi} N(r) \right], \quad (19)$$

where K is a constant and $N(r)$ is a number of particles inside the radius r , $0 < r < R$; here and later all macroscopic quantities are taken per unit length of a container (see also Appendix).

Our general solution (19) represents a superposition of a vortex line of intensity K and rigid-body rotation of cylindrical layers with a local angular velocity $\Omega(r) = \Gamma\nu(r)/2$. The angular momentum of the liquid $L = L_z$ can be found as

$$L = m \int d^2r \nu(r) r u(r). \quad (20)$$

Using eq. (19), we derive

$$L = mNK + \frac{m\Gamma}{4\pi} N^2, \quad (21)$$

where $N = N(R)$ is the total particle number. In the absence of Γ , as usual, the bosonic quantization $L = N\hbar s$, where s is an integer, determines $K_s = s\hbar/m$. The additional part of the angular momentum does not carry any singularity at $r \rightarrow 0$.

For a vortex-free case, $K = 0$, we have from $L = N\hbar s$

$$\Gamma = \frac{4\pi\hbar}{mN} s, \quad (22)$$

which corresponds to rotation with local linear velocity $v_b(r) = \Gamma\nu(r)r/2$.

4. Solutions with and without a vortex

To move further, we assume for simplicity that the functional \overline{W} does not contain gradients of n and can be expanded around the unperturbed density of non-rotating liquid, n_0 ,

$$\overline{W} = \overline{W}\{n_0\} + \frac{1}{2} m^2 g \int d^2r [n(r) - n_0]^2. \quad (23)$$

With this functional, the balance equation (17) takes the form (see Appendix for derivation)

$$\frac{mu^2}{r} = m^2 g \frac{d\nu}{dr} + \frac{\hbar^2}{8m} \frac{d}{dr} \left[\frac{1}{\nu^2} \left(\frac{d\nu}{dr} \right)^2 - \frac{2}{\nu r} \frac{d}{dr} \left(r \frac{d\nu}{dr} \right) \right]. \quad (24)$$

The compressibility g defines the speed of sound, $c = \sqrt{mgn_0}$. It is convenient to introduce dimensionless variables and express velocities in units of c , energies in units mc^2 , lengths in units of $r_0 = \hbar/mc$, and density in units of n_0 . Then the only remaining parameter is Γ [in usual units $\Gamma \Rightarrow \hbar n_0 \Gamma / (mc^2)$]. For typical quantum liquids, the length r_0 corresponds to atomic distances, while the macroscopic container has $R \gg 1$. At the same time, for reasonable angular velocity, the rotation is subsonic, $\Gamma R < 1$, which means $\Gamma \ll 1$ in reduced units.

As follows from eq. (19), we can have two types of solutions for eq. (24) with a regular behavior of density $\nu(r)$ at the axis, $r \rightarrow 0$. For the vortex line solution, $K \neq 0$, at small distances from the axis the velocity is $u \approx K/r$, while the density goes to zero, $\nu(r) \propto r^{2|K|} + \mathcal{O}(r^{4|K|+2})$, and its behavior is determined by the quantum term (11). Here Γ influences only higher terms of the expansion of $\nu(r)$, the radius of the vortex core does not noticeably depend on Γ and has the magnitude of the order of one (the distance where we can equate contributions to the energy from compressibility and from quantum effects). Such results for quantized vortices are well known [1, 36]; they are modified by rotation only at large distances from the center. Below we will be interested in the solutions of the second type, where the vortex singularity is absent, $K = 0$.

At $K = 0$, the velocity and density satisfying eqs. (24) and (19) can be expressed by a well converging series in powers of a parameter $\xi = \nu_0 \Gamma^2 R^2$:

$$\nu(r) = \nu_0 \left\{ 1 + \frac{\xi}{8} \frac{r^2}{R^2} + \frac{\xi^2}{128} \frac{r^4}{R^4} + \dots \right\}, \quad (25)$$

$$u(r) = \frac{1}{2} \nu_0 \Gamma r \left\{ 1 + \frac{\xi}{16} \frac{r^2}{R^2} + \frac{\xi^2}{384} \frac{r^4}{R^4} + \dots \right\}. \quad (26)$$

The central density $\nu(0) \equiv \nu_0$ is determined by the particle number conservation being related to the unperturbed value $n_0 = 2u(R)/(R\Gamma) \equiv 1$ through a similar series,

$$\nu_0^{-1} = 1 + \frac{\xi}{16} + \frac{\xi^2}{384} + \dots \quad (27)$$

It is easy to estimate that in such a state the contributions to energy per particle from the kinetic term and from compressibility are, by order of magnitude, $\sim \Gamma^2 R^2$ and $\sim \Gamma^4 R^4$, respectively, whereas the quantum potential brings in only a small contribution $\sim \Gamma^2$. The angular momentum, eq. (21), equals $L = N\hbar s$, where $s = \Gamma R^2/4$; at $\Gamma R \sim 1$, this value $s \gg 1$. In the case of the external cranking with angular velocity Ω , the state with a certain value $\Gamma \neq 0$ is getting energetically favorable at a critical value

$$\Omega_c(\Gamma) = \frac{E(\Gamma) - E(0)}{L(\Gamma)} \approx \frac{\Gamma}{4}. \quad (28)$$

For the minimum value $\Gamma_{\min} = \Gamma(s = 1) \approx 4/R^2$, we have $\Omega_c^{\min} \approx R^{-2}$. This type of rotation can emerge at a smaller (by a factor of $\ln R$) angular velocity than the creation of usual quantized vortices. However, the probability of such an excitation, for example by an accidental nonuniformity of the walls of the container [37], is considerably lower, while the observation at $\Gamma R < 1$ is difficult.

5. Small amplitude oscillations

Now we will construct the low-lying excited states corresponding to small oscillations of the density and transverse velocity around the classical solution [eqs. (25-27)]. We introduce the exciton operators, $\hat{\mathbf{v}}(\mathbf{r}) \equiv \mathbf{v} - \mathbf{u}$ and $\hat{n}(\mathbf{r}) \equiv n - \nu$. The linearization of the equations of motion (15) in the model (23) gives

$$[\hat{\mathbf{v}}, H] = -i\vec{\nabla}(\mathbf{u} \cdot \hat{\mathbf{v}} + \hat{P}) + i[(\mathbf{u}\hat{n} + \hat{\mathbf{v}}\nu) \times \vec{\Gamma}], \quad (29)$$

or, in cylindrical coordinates,

$$[\hat{v}_r, H] = -i\frac{\partial}{\partial r}(u\hat{v}_\varphi + \hat{P}) + i\Gamma(\hat{v}_\varphi\nu + u\hat{n}), \quad (30)$$

and

$$[\hat{v}_\varphi, H] = -\frac{i}{r}\frac{\partial}{\partial \varphi}(u\hat{v}_\varphi + \hat{P}) - i\Gamma\hat{v}_r\nu. \quad (31)$$

Here the operator was introduced,

$$\hat{P} = \hat{n} - \frac{1}{4} \left\{ \nabla^2 \left(\frac{\hat{n}}{\nu} \right) - \frac{\vec{\nabla}\nu}{\nu} \cdot \vec{\nabla} \left(\frac{\hat{n}}{\nu} \right) \right\}, \quad (32)$$

and, according to eq. (12),

$$\Gamma\hat{n} = \frac{1}{r} \left\{ \frac{\partial}{\partial r}(r\hat{v}_\varphi) - \frac{\partial \hat{v}_r}{\partial \varphi} \right\}. \quad (33)$$

The continuity equation,

$$[\hat{n}, H] = -i\frac{u}{r}\frac{\partial \hat{n}}{\partial \varphi} - \frac{1}{r} \left\{ \frac{\partial}{\partial r}(r\nu\hat{v}_r) + i\nu\frac{\partial \hat{v}_\varphi}{\partial \varphi} \right\}, \quad (34)$$

follows from the equations of motion. Finally, with use of eq. (18), we can write down the equations of motion in a symmetric form,

$$[\hat{v}_r, H] + i\frac{u}{r}\frac{\partial \hat{v}_r}{\partial \varphi} - 2i\frac{u}{r}\hat{v}_\varphi = -i\frac{\partial \hat{P}}{\partial r}, \quad (35)$$

$$[\hat{v}_\varphi, H] + i\frac{u}{r}\frac{\partial \hat{v}_\varphi}{\partial \varphi} + i\Gamma\nu\hat{v}_r = -\frac{i}{r}\frac{\partial \hat{P}}{\partial \varphi}. \quad (36)$$

At azimuthal symmetry of the ground state, the elementary Bose-excitations can be characterized by a certain angular momentum $L_z = \hbar\ell$. The quantization goes with expansion of our fields over corresponding creation, $A_{k\ell}^\dagger$, and annihilation, $A_{k\ell}$, operators, where the additional subscript k labels consecutive modes for given ℓ . The expansion can be written as

$$\begin{pmatrix} \hat{v}_r(\mathbf{r}) \\ \hat{v}_\varphi(\mathbf{r}) \\ \hat{n}(\mathbf{r}) \end{pmatrix} = \sum_{\ell} \frac{e^{i\ell\varphi}}{\sqrt{2\pi}} \begin{pmatrix} \hat{v}_r^{(\ell)}(r) \\ \hat{v}_\varphi^{(\ell)}(r) \\ \hat{n}^{(\ell)}(r) \end{pmatrix} = \sum_{k\ell} \frac{e^{i\ell\varphi}}{\sqrt{4\pi}} \begin{pmatrix} \left\{ R_{k\ell}(r)A_{k\ell} + R_{k-\ell}^*(r)A_{k-\ell}^\dagger \right\} \\ i \left\{ \Phi_{k\ell}(r)A_{k\ell} - \Phi_{k-\ell}^*(r)A_{k-\ell}^\dagger \right\} \\ i \left\{ F_{k\ell}(r)A_{k\ell} - F_{k-\ell}^*(r)A_{k-\ell}^\dagger \right\} \end{pmatrix}. \quad (37)$$

Here the unknown radial form-factors of excitations, $R_{k\ell}$, $\Phi_{k\ell}$, and $F_{k\ell}$, are the matrix elements between the ground state and excited one-quantum state with quantum numbers k, ℓ and energy $E_{k\ell}$. They satisfy the set of linear differential equations,

$$\left(E_{k\ell} - \ell \frac{u}{r}\right) R_{k\ell} + 2 \frac{u}{r} \Phi_{k\ell} = \frac{dP_{k\ell}}{dr}, \quad (38)$$

$$\left(E_{k\ell} - \ell \frac{u}{r}\right) \Phi_{k\ell} + \Gamma \nu R_{k\ell} = \frac{\ell}{r} P_{k\ell}, \quad (39)$$

$$\Gamma F_{k\ell} = \frac{d\Phi_{k\ell}}{dr} + \frac{\Phi_{k\ell}}{r} - \frac{\ell}{r} P_{k\ell}. \quad (40)$$

The right hand side quantities $P_{k\ell}$ are related to the matrix elements of density:

$$P_{k\ell} = F_{k\ell} - \frac{1}{4} \left[\nabla_\ell^2 - \frac{1}{\nu} \frac{d\nu}{dr} \frac{d}{dr} \right] \frac{F_{k\ell}}{\nu}, \quad (41)$$

where the differential operator ∇_ℓ^2 is defined as

$$\nabla_\ell^2 \equiv \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{\ell^2}{r^2}. \quad (42)$$

To analyze this set of equations, it is convenient to introduce the notations

$$\mathcal{E}_{k\ell}(r) = E_{k\ell} - \ell \frac{u}{r}, \quad D_{k\ell}(r) = \mathcal{E}_{k\ell}^2 - 2\Gamma \frac{\nu u}{r}, \quad (43)$$

where $\mathcal{E}_{k\ell}(r)$ can be interpreted as “excitation energy” in the frame rotating together with the layer $r = \text{const.}$ The matrix elements of the velocity components can be presented as

$$R_{k\ell}(r) = D_{k\ell}^{-1} \left(\mathcal{E}_{k\ell} P'_{k\ell} - 2\ell \frac{u}{r^2} P_{k\ell} \right), \quad (44)$$

and

$$\Phi_{k\ell}(r) = D_{k\ell}^{-1} \left(\mathcal{E}_{k\ell} \frac{\ell}{r} P_{k\ell} - \Gamma \nu P'_{k\ell} \right); \quad (45)$$

here and below the prime indicates the radial derivative d/dr . Returning to the definition (40), we obtain a differential equation for $F_{k\ell}$:

$$\begin{aligned} & D_{k\ell} F_{k\ell} + \nu \nabla_\ell^2 P_{k\ell} \\ &= \left\{ -\frac{\ell^2}{r^2} \left(\nu - \frac{2u}{r\Gamma} \right) + \frac{\ell}{r} \left(\mathcal{E}'_{k\ell} \Gamma^{-1} - \frac{D'_{k\ell}}{D_{k\ell}} \mathcal{E}_{k\ell} \right) \right\} P_{k\ell} - \left(\nu' - \nu \frac{D'_{k\ell}}{D_{k\ell}} \right) P'_{k\ell}. \end{aligned} \quad (46)$$

As seen from eq. (41), the right hand side of (46) is the third order differential operator acting on $F_{k\ell}$; the coefficients of the operator contain only small terms of the expansions (25) and (26). Therefore we can act iteratively.

Here we will limit ourselves by the main order assuming

$$\nu \approx \nu_0 \approx 1, \quad \frac{u}{r} \approx \frac{\Gamma}{2}, \quad \mathcal{E}_{k\ell} \approx E_{k\ell} - \frac{1}{2} \ell \Gamma = \text{const.} \quad (47)$$

Then eq. (46) is reduced to

$$\left(\mathcal{E}_{k\ell}^2 - \Gamma^2 + \nabla_\ell^2 - \frac{1}{4} \nabla_\ell^4 \right) F_{k\ell}^{(0)}(r) = 0. \quad (48)$$

The regular at the origin solution is a superposition of cylindrical functions

$$F_{k\ell}^{(0)}(r) = b_{k\ell}^{(+)} J_\ell(q_{k\ell}^{(+)} r) + b_{k\ell}^{(-)} I_\ell(q_{k\ell}^{(-)} r), \quad (49)$$

where the wave numbers $q_{k\ell}^{(\pm)}$ are related to energy $E_{k\ell}$ by a dispersion relation

$$\mathcal{E}_{k\ell}^2 = \Gamma^2 \pm q_{k\ell}^{(\pm)2} + \frac{1}{4} q_{k\ell}^{(\pm)4}, \quad (50)$$

or, selecting the appropriate roots,

$$q_{k\ell}^{(\pm)2} = 2\left(\sqrt{1 + \mathcal{E}_{k\ell}^2 - \Gamma^2} \mp 1\right). \quad (51)$$

In the same approximation we obtain for the function (41):

$$P_{k\ell}^{(0)}(r) = \frac{1}{4} \left\{ b_{k\ell}^{(+)} q_{k\ell}^{(-)2} J_\ell(q_{k\ell}^{(+)} r) - b_{k\ell}^{(-)} q_{k\ell}^{(+2)} I_\ell(q_{k\ell}^{(-)} r) \right\}. \quad (52)$$

We came to the direct generalization of the standard phonon spectrum for the case of the frozen vortex-like motion. There is now a small gap Γ in the phonon spectrum of transverse oscillations. The gap is determined by the magnitude of $\text{curl } \mathbf{v}$.

The coefficients $b_{k\ell}^{(\pm)}$ of the superposition (49), as well as quantization of the eigenvalues $\mathcal{E}_{k\ell}$, are determined by the normalization of the solutions and the boundary conditions at $r = R$. For our problem of vibrations on the background of a classical solution with almost constant density, the natural boundary conditions in the accepted approximation are those of vanishing normal components of the velocity, $R_{k\ell}$, and of the gradient of density, $F'_{k\ell}$. A detailed discussion of boundary conditions can be found in Ref. [38]. With the aid of eqs. (41,49) and (52), we find, up to a normalization constant $C_{k\ell}$,

$$b_{k\ell}^{(+)} = C_{k\ell} I'_\ell(q_{k\ell}^{(-)} R), \quad b_{k\ell}^{(-)} = -C_{k\ell} J'_\ell(q_{k\ell}^{(+)} R). \quad (53)$$

The energy quantization is given by ($z_\pm = q_{k\ell}^{(\pm)} R$)

$$J'_\ell(z_+) I'_\ell(z_-) \frac{z_+^2 + z_-^2}{R^2} - \Gamma \frac{\ell}{R^3} \left[z_-^2 J_\ell(z_+) I'_\ell(z_-) + z_+^2 I_\ell(z_-) J'_\ell(z_+) \right] = 0. \quad (54)$$

As it follows from eq. (51) that $q^{(-)2} - q^{(+2)} = 4$, we need $q^{(-)} \geq 1$. For a macroscopic vessel, $R \gg 1$, and $\ell < R$, the cylindrical functions I_ℓ and I'_ℓ in the last equation can be taken in asymptotics. This leads to

$$\mathcal{E}_{k\ell} q_{k\ell}^{(-)} (q_{k\ell}^{(+2)} + q_{k\ell}^{(-)2}) J'_\ell(z_+) - \Gamma \frac{\ell}{R} (q_{k\ell}^{(-)3} J_\ell(z_+) + q_{k\ell}^{(+2)}) = 0. \quad (55)$$

In the irrotational case, $\Gamma = 0$, or for $\ell = 0$, the dispersion equation (55) determines the usual hydrodynamic spectrum $z = z_{k\ell}^\circ$, where $J'_\ell(z_{k\ell}^\circ) = 0$. Assuming a small shift $\delta z_{k\ell}$ of eigenvalues, we find (omitting subscripts k, ℓ)

$$\frac{\delta z}{z^\circ} \approx \frac{4R^2 + (z^\circ)^2}{4R^2 + 2(z^\circ)^2} \sqrt{\frac{\xi}{\xi + (z^\circ)^2 + (z^\circ)^4/4R^2}} \frac{\ell}{\ell^2 + (z^\circ)^2}, \quad (56)$$

where the parameter ξ was used earlier, eq. (25). This result is valid for short wavelengths, when $q^{(+)} \sim q^{(-)} \sim 1$, $z^\circ \sim R \sim \ell$, and $\delta z/z^\circ \sim \Gamma/\ell \ll 1$, but also in the long wavelength limit, $q^{(+)} \ll 1$, if ℓ is sufficiently large, $1 \ll \ell \sim q^{(+)} R \ll R$. In the last case it is applicable even for $\xi \sim (z^\circ)^2$ since $\delta z/z^\circ \sim \ell^{-1} \ll 1$. For excitations

with small values of quantum numbers $\ell, k \sim 1$ and $\Gamma \sim 1/R$, the displacements of the roots is significant, and one has to return to the general eq. (54). In that case, $\Gamma \sim q^{(+)}$, and the presence of the gap in the spectrum (50) is essential.

The simplest way to finding the normalization constants is based on the orthogonality conditions following from the set of equations (38-40),

$$\int_0^R r dr \{ R_{k'\ell}^{(0)} R_{k\pm\ell}^{(0)} \pm \Phi_{k'\ell}^{(0)} \Phi_{k,\pm\ell}^{(0)} \pm F_{k'\ell}^{(0)} P_{k,\pm\ell}^{(0)} \} = \frac{1 \pm 1}{2} \Theta_{k\ell} \delta_{kk'}, \quad (57)$$

where the right hand side quantities $\Theta_{k\ell}$, after some algebra using the equations and boundary conditions, can be expressed as

$$\Theta_{k\ell} = - \frac{\Gamma \ell}{D_{k\ell} \mathcal{E}_{k\ell}} [P_{k\ell}^{(0)}(R)]^2 + \frac{2\mathcal{E}_{k\ell}^2}{D_{k\ell}} \int_0^R dr r F_{k\ell}^{(0)} P_{k\ell}^{(0)}. \quad (58)$$

On the other hand, conditions (57) and operator expansions (37) show that

$$A_{k\ell} = \frac{\sqrt{2}}{\Theta_{k\ell}} \int_0^R dr r \{ R_{k\ell}^{(0)} \hat{v}_r^{(\ell)} - i \Phi_{k\ell}^{(0)} \hat{v}_\varphi^{(\ell)} - i P_{k\ell}^{(0)} \hat{n}^{(\ell)} \}. \quad (59)$$

The quantization according to the Bose-statistics, $[A_{k\ell}, A_{k'\ell'}^\dagger] = \delta_{kk'} \delta_{\ell\ell'}$, together with the original rules of quantum hydrodynamics (7) transformed in our problem to eq. (14), leads in this approximation to

$$\frac{4}{\Theta_{k\ell}^2} \int_0^R dr r \{ \Gamma R_{k\ell}^{(0)} \Phi_{k\ell}^{(0)} + \mathcal{E}_{k\ell} [(R_{k\ell}^{(0)})^2 + (\Phi_{k\ell}^{(0)})^2] \} = 1. \quad (60)$$

The explicit calculation of the integral (60) and comparison to eq. (58) determines the simple normalization condition,

$$\Theta_{k\ell} = 2\mathcal{E}_{k\ell}. \quad (61)$$

The same result can be derived by the direct diagonalization of the linearized Hamiltonian. We will not write down a cumbersome expression for the normalization constants $C_{k\ell}$ that can be derived from eqs. (58) and (61) with the aid of the solutions (49) and (52). Here we show only the result for a pure hydrodynamic spectrum that becomes exact for $\ell = 0$, when $b_{k\ell}^{(-)} \rightarrow 0$, while

$$(b_{k\ell}^{(+)})^2 \approx 8D_{k\ell} \left\{ (q_{k\ell}^{(-)})^2 J_\ell^2(q_{k\ell}^{(+)} R) \left[\mathcal{E}_{k\ell} R^2 \left(1 - \frac{\ell^2}{(q_{k\ell}^{(+)} R)^2} \right) - \frac{\Gamma \ell (q_{k\ell}^{(+)})^2}{4\mathcal{E}_{k\ell}^2} \right] \right\}^{-1}. \quad (62)$$

In spite of the presence of the gap in the energy spectrum (50), we still obtain in the transition to an infinite system, $R \rightarrow \infty$, a vanishing interaction strength with the long-wavelength, $q^{(+)} \rightarrow 0$, phonons. Indeed, in agreement with general arguments, we see from eq. (62) that

$$b_{k\ell}^{(+)} \sim \sqrt{D_{k\ell}} \propto q_{k\ell}^{(+)}. \quad (63)$$

It is easy to see also that the off-diagonal matrix elements of the local density are small being at $\Gamma \sim q^{(+)} \sim 1/R$ of the order $N^{-1/2}$ as it should be for collective excitations,

$$\langle k\ell | \hat{n}(r) | 0 \rangle \sim \sqrt{\frac{q_{k\ell}^{(+)}}{N}} \frac{J_\ell(q_{k\ell}^{(+)} r)}{J_\ell(q_{k\ell}^{(+)} R)}. \quad (64)$$

This justifies the linearization similar to the random phase approximation.

Our energy spectrum (50) formally allows for two types of excitations with phonon angular momentum ℓ and energies

$$E_{k\ell}^{\pm} = \frac{\ell\Gamma}{2} \pm \sqrt{\Gamma^2 + (q_{k\ell}^{(+)})^2 + (q_{k\ell}^{(+)})^4/4}. \quad (65)$$

In the long wavelength limit, the Coriolis interaction, $-\ell\Omega$, of the “minus” phonon with $\ell > 2$ is so strong that the creation of such a phonon becomes energetically favorable at sufficiently large angular velocity, namely if $\Gamma > 2\{[(q_{k\ell}^{(+)})^2 + (q_{k\ell}^{(+)})^4/4]/(\ell^2 - 4)\}$. However, this would correspond to the supersonic velocity of rotation at the boundary, $\Gamma R/2 > 1$, when the approximation used above is invalid and the spatial non-uniformity is essential. In our approximation we have to take into account only the “plus” branch $E_{k\ell}^{+}$.

6. Discussion

The study of rotating helium II has a long history. Contrary to the predictions of Landau theory[30], it appeared that below the λ -point the superfluid components behaved in rotation as a classical fluid. This was shown by measuring the shape of the free surface meniscus of rotating helium that revealed a classical profile [39, 40, 41, 42, 43, 44]. The measurement of the angular momentum of rotating helium II was also found to correspond to the rigid body rotation of the liquid[45, 46, 47]. As is well known, this paradox was solved by the Onsager-Feynman theory of quantized vortices[3]. The critical velocity for the vortex nucleation is $\Omega_s = (\hbar/mR^2) \ln(R/a)$, where R is the radius of the vessel and a is the radius of the vortex core. In the aforementioned experiments the speed of rotation was much higher than Ω_s so that the vortex lines formed a lattice that on average imitated the rigid-body rotation [3].

On the other hand, we cannot exclude the manifestation of “rigid-body” rotation in several reported experiments. For example, the interpretation of some data in [22] as Landau state does not seem unambiguous. In this paper the helium was contained in a torus resonator filled with slip rings and brushes so that the normal-fluid component in rotation was locked to the resonator by its viscous interaction. The relative velocity of the normal fluid and superfluid was determined by measuring the frequency difference of fourth-sound modes which run around the resonator in both directions. The dependence of the superfluid angular velocity on the normal-fluid angular velocity had the form of a hysteretic curve. The reversible part of this curve was attributed to the vortex-free potential flow (Landau state). Plausibly, this reversible part could be as well attributed to the “rigid” velocity field, since for the multiple-connected geometry the contribution from such a rotation of superfluid (if it does exist) cannot be distinguished from the contribution of irrotational potential motion.

It is also appropriate to mention here the paper [48], where the angular momentum of superfluid helium was measured after the rotating helium bucket has been stopped. The resulting angular momentum was smaller than the angular momentum of rotating

bucket, and the authors came to a conclusion that the reason for this was the formation of the long-lived macroscopic rotational state that accepted part of the angular momentum. As the authors pointed out, this fact could not be explained neither by the Landau two-fluid theory (in a simply connected region, the only solution for potential flow of incompressible fluid is $v = 0$) nor by Onsager-Feynman vortex states. In this connection it is worthwhile to remember the so called Lin's hydrodynamics[49], where, to our knowledge, for the first time the conception of rotational superfluid flow has been proposed. Lin has suggested that the superfluid might have very small nonzero viscosity and would thus reach a steady state of uniform rotation in which boundary slip would cause the superfluid to rotate more slowly than the normal fluid at low velocities. Lin has applied his approach for the explanation of the variety of helium-II experiments. In particular, the above mentioned experiment [48] has found a satisfactory explanation within his theory[50].

For our velocity field ((19) with $K = 0$), it is easy to calculate critical velocities for slow rotation assuming that the density is constant over the volume. For the rotation in a circular cylinder of radius R we obtain, by minimizing the free energy in a rotating frame, the critical angular velocities for vortex-free rotation $\Omega_b = \Omega_0(2p+1)$, where $p = 0, 1, 2, \dots$. Here $\Omega_0 = \hbar/mR^2$, and (see (22)) $s = 0$ for $\Omega < \Omega_0$, $s = 1$ for $\Omega_0 < \Omega < 3\Omega_0$, $s = 2$ for $3\Omega_0 < \Omega < 5\Omega_0$ and so on. Comparing this with the critical velocity for the appearance of the first vortex Ω_s , we see that the first critical velocity for vortex-free rotation Ω_b is approximately ten times smaller than the vortex critical velocity Ω_s . In general, our critical velocities are on the order of Ω_0 which even for $R \approx 1$ mm are rather small, about 0.01 rad/s.

Now that the instrumentation methods are greatly advanced as compared to those used in early experiments, a possible clear-cut experiment for a search of the frozen rotation would be the measuring of the free surface of rotating helium with the speed of rotation being much less than the critical velocity for the vortex nucleation. Other possibility is the direct investigation of the flow pattern below the λ -point with the aid of some visualization technique (see[51] and references therein). It is also worthwhile to note the paper[52] where the circularly polarized third-sound wave modes were used to drive persistent flow states in a superfluid helium film inside the resonator. As was pointed in this paper, this method could, in principle, be used to find the profile of the velocity field.

One can also point out a possible connection of our solution to the effects emerging in the flow of helium-II in porous glass with the size 1-10 nm of the pores [53], narrow slits[54], in small tubes [55], and submicron channels [56, 57]. In particular, as was reported in [56], the critical velocities for the flow of superfluid ^4He through a submicron aperture appeared to be much smaller than it would be necessary for a vortex ring nucleated at the wall.

An interesting question of experimental manifestations of a "rigid-body" rotation of quantum Bose-fluids other than liquid helium requires special consideration and is beyond the scope of the present paper.

7. Conclusion

In this article we made an attempt to vindicate the old operator formulation of quantum hydrodynamics due to Landau [30] and Geilikman [31]. This formulation does not lead to any unphysical states of the type of the "fall onto center" and allows one, in the simplest cylindrical geometry, to write down and solve the operator equations of motion for a quantum liquid. An example of the explicit solution was found in the form of the classical background and collective oscillatory modes. The solution in general contains a superposition of a vortex line of intensity K that should be quantized through its relation to the angular momentum and a rigid-body rotation with a local angular velocity $\Omega(r) = \Gamma\nu(r)/2$ due to the velocity curl "frozen" into the density of the liquid.

We considered in detail the solutions without a vortex singularity, $K = 0$. The calculations are rather long but straightforward. On top of such a solution we superimpose small oscillations of transverse components of the velocity and related oscillations of density and linearize the equations of motion. For the spatial function we get a complete set of modes with orbital momentum ℓ and the radial dependence determined by the combination of cylindrical functions $J_\ell(kr)$ and $I_\ell(kr)$ that is determined by the boundary conditions.

Now, when the new types of quantum liquids are under experimental scrutiny, the search for diverse manifestations of rotational superfluid flow is at the frontiers of current interest. In any case, it seems that the conventional consideration of the vortices as the only long lived objects in rotating quantum liquids is too restrictive.

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Appendix. Minimization of the density functional.

To find the ground state based on the rotational velocity profile,

$$u(r) = \frac{\Gamma}{2\pi r} N(r), \quad (66)$$

we have to minimize the functional depending on $\nu(r)$ and its gradient $d\nu/dr$. The functional contains kinetic energy \mathcal{K} , potential (interaction) energy \mathcal{W} , and quantum potential \mathcal{Q} . Let us consider these terms one by one.

The kinetic term is

$$\mathcal{K} = \frac{m}{2} \int d^2r \nu u^2. \quad (67)$$

Here we minimize directly taking into account that we restrict ourselves by the class of states, where eq. (66) is valid,

$$\delta\mathcal{K} = \frac{m\Gamma^2}{8\pi^2} \int \frac{d^2r}{r^2} \left\{ \delta\nu(r) N^2(r) + 2\nu(r) N(r) \delta N(r) \right\}. \quad (68)$$

Here, using the step function $\Theta(x)$,

$$\delta N(r) = \int d^2r' \theta(r - r') \delta\nu(r'), \quad (69)$$

and changing notation in the double integral $r \leftrightarrow r'$,

$$\delta\mathcal{K} = \frac{m\Gamma^2}{8\pi^2} \int d^2r \left\{ \frac{N^2(r)}{r^2} + 2 \int \frac{d^2r'}{r'^2} \theta(r' - r) \nu(r') N(r') \right\} \delta\nu(r). \quad (70)$$

The potential term,

$$\mathcal{W} = \frac{1}{2} m^2 g \int d^2r (\nu - n_0)^2, \quad (71)$$

is trivial,

$$\delta\mathcal{W} = m^2 g \int d^2r (\nu - n_0) \delta\nu(r). \quad (72)$$

The quantum term,

$$\mathcal{Q} = \frac{\hbar^2}{8m} \int d^2r \frac{(\nabla\nu)^2}{\nu}, \quad (73)$$

depends both on ν and $\nabla\nu$. The variation goes as in the Lagrangian mechanics,

$$\delta\mathcal{Q} = \frac{\hbar^2}{8m} \int d^2r \delta\mathcal{L}(\nu, \nabla\nu) = \frac{\hbar^2}{8m} \int d^2r \left\{ \frac{\partial\mathcal{L}}{\partial\nu} \delta\nu + \frac{\partial\mathcal{L}}{\partial(\nabla\nu)} \cdot \delta(\nabla\nu) \right\}. \quad (74)$$

Here

$$\frac{\partial\mathcal{L}}{\partial\nu} = - \frac{(\nabla\nu)^2}{\nu^2}, \quad (75)$$

$$\frac{\partial\mathcal{L}}{\partial(\nabla\nu)} = \frac{2}{\nu} (\nabla\nu \cdot \delta(\nabla\nu)). \quad (76)$$

Since the symbols δ and ∇ commute, eq. (76) gives

$$2\nabla \cdot \left(\frac{\nabla\nu}{\nu} \delta\nu \right) - 2 \left(\nabla \cdot \frac{\nabla\nu}{\nu} \right) \delta\nu = 2\nabla \cdot \left(\frac{\nabla\nu}{\nu} \delta\nu \right) - 2 \frac{\nabla^2\nu}{\nu} + 2 \frac{(\nabla\nu)^2}{\nu^2}. \quad (77)$$

Combining eqs. (75) and (76), we obtain

$$\delta \mathcal{Q} = \frac{\hbar^2}{8m} \int d^2r \left\{ \frac{(\nabla \nu)^2}{\nu^2} - 2 \frac{\nabla^2 \nu}{\nu} \right\} \delta \nu + \delta \mathcal{Q}_S, \quad (78)$$

where we have also a surface term,

$$\delta \mathcal{Q}_S = \frac{\hbar^2}{4m} \int d^2r \nabla \cdot \left(\frac{\nabla \nu}{\nu} \delta \nu \right). \quad (79)$$

If we forget about the surface contribution, the Lagrange equation is

$$\begin{aligned} m^2 g(\nu - n_0) + \frac{\hbar^2}{8m} \left\{ \frac{(\nabla \nu)^2}{\nu^2} - 2 \frac{\nabla^2 \nu}{\nu} \right\} \\ = - \frac{m \Gamma^2}{8\pi^2} \left\{ \frac{N^2(r)}{r^2} + 2 \int \frac{d^2 r'}{r'^2} \theta(r' - r) \nu(r') N(r') \right\}. \end{aligned} \quad (80)$$

Taking the gradient d/dr in eq. (80) and using again eq. (66), we come to eq. (24).